Section 1 Problems

1. Write out the first 5 terms of the sequences defined as follows. Let the initial value of n be zero unless there is a reason why it cannot be. Simplify as much as possible.

   a. $a_n = \dfrac{(n+1)!}{n!}$ d. $a_n = \ln(\ln n)$

   b. $a_n = \dfrac{\cos(n\pi)}{n}$ e. $a_n = \dfrac{3^n}{4^{n+1}}$

   c. $a_n = \sqrt[n]{n}$

2. Give an expression for the general term of the following sequences.

   a. $2, 4, 6, 8, \ldots$ c. $1, 4, 27, 256, \ldots$

   b. $-1, 1, \frac{-1}{2}, \frac{1}{6}, \frac{-1}{24}, \frac{1}{120}, \ldots$

3. Which of the sequences in Problems 1 and 2 converge? To what value do they converge?

4. For what values of x does the sequence defined by $a_n = \dfrac{x^n}{n!}$ converge? What value does it converge to?

5. Evaluate s_5 and s_{10} for the following series. If you can, make a conjecture for the sum of the series.

   a. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ c. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

   b. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \cdots$

In problems 6-15, determine whether the given series definitely converges, definitely diverges, or its convergence cannot be determined based on information from this section. Give a reason to support your answer.

6. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

7. $2 - 4 + 8 - 16 + 32 - \cdots$

8. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$

9. $\displaystyle\sum_{n=1}^{\infty} \cos\left(\tfrac{1}{n}\right)$

10. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\tfrac{1}{n}\right)$

11. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$

12. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{\pi}{e}\right)^n$

13. $\displaystyle\sum_{n=1}^{\infty} \sin(n)$

14. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{3^n + n}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

In problems 16-23, find the sum of the convergent series.

16. $\displaystyle\sum_{n=0}^{\infty} \left(\tfrac{2}{5}\right)^n$

17. $\displaystyle\sum_{n=0}^{\infty} 2 \cdot \left(\tfrac{1}{8}\right)^n$

18. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{8^{n+2}}$

19. $\displaystyle\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n}$

20. $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n}$

21. $\displaystyle\sum_{n=10}^{\infty} \left(\tfrac{3}{4}\right)^n$

22. $\displaystyle\sum_{n=1}^{\infty} \tfrac{5}{3^n}$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{5 + 3^n}{4^n}$

24. Represent the following repeating decimals as fractions of integers.

   a. $0.777\overline{7}$ c. $0.317\overline{317}$

b. $0.8\overline{2}$ d. $2.43\overline{838}$

25. Prove that $0.\overline{9}=1$.

26. A superball is dropped from a height of 6 ft and allowed to bounce until coming to rest. On each bounce, the ball rebounds to 4/5 of its previous height. Find the total up-and-down distance traveled by the ball.

27. Repeat exercise 26 for a tennis ball that rebounds to 1/3 of its previous height after every bounce. This time suppose the ball is dropped from an initial height of 1 meter.

28. Repeat exercise 26 for a bowling ball that rebounds to only 1/100 of its previous height after every bounce.

29. The St. Ives nursery rhyme goes as follows:

   "As I was walking to St. Ives / I met a man with 7 wives / Each wife had seven sacks / Each sack had seven cats / Each cat had seven kits / Kits, cats, sacks, wives / How many were going to St. Ives?"

   Use sigma notation to express the number of people and things (kits, cats, etc.) that the narrator encountered. Evaluate the sum.

30. Evaluate the following sum:

$$\sum_{\substack{k \text{ is divisible} \\ \text{only by 2 or 3}}}^{\infty} \frac{1}{k}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{12}\cdots$$

   (Hint: This series can be regrouped as an infinite series of geometric series.)

31. Evaluate the sum $\sum_{n=1}^{\infty}\frac{n}{2^n}$.

   (Hint: Rewrite all the fractions as unit fractions. For example, rewrite the term $\frac{3}{2^3}$ as $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$. Then regroup to form an infinite series of geometric series.)

32. Generalize the result of Problem 31 to give the sum of $\sum_{n=1}^{\infty}\frac{n}{r^n}$, where $|r|<1$.

Another type of series is the so-called "telescoping" series. An example is

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$$
$$=\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cancel{\cdots}$$

The series is called a "telescoping" series because it collapses on itself like a mariner's telescope.

33. a. Find an expression for the partial sum s_n of the telescoping series shown above.

    b. Compute the sum of the example telescoping series.

In problems 34-37, find the sums of the telescoping series. You may have to use some algebraic tricks to express the series as telescoping.

34. $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$

35. $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$

36. $\sum_{n=2}^{\infty}\ln\left(\frac{n}{n+1}\right)$

37. $\sum_{n=0}^{\infty}\left(\arctan(n+1)-\arctan(n)\right)$

38. Give an example of two *divergent* series $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ such that $\sum_{n=1}^{\infty}\frac{a_n}{b_n}$ converges.

For problems 39-44 indicate whether the statement is True or False. Support your answer with reasons and/or counterexamples.

39. If $\lim_{n\to\infty}a_n=0$, then $\sum_{n=1}^{\infty}a_n$ converges.

40. If $\sum_{n=1}^{\infty}a_n$ converges, then $\lim_{n\to\infty}a_n=0$.

41. If $\sum_{n=1}^{\infty}a_n$ converges, then $\lim_{n\to\infty}s_n=0$.

42. If $\sum_{n=1}^{\infty}a_n$ converges, then $\sum_{n=0}^{\infty}\frac{1}{a_n}$ converges.